Vector fields on surfaces

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Vector fields

Definition

If Σ is a surface (in $\mathbb{R}^n$), then a vector field on Σ is a choice of a tangent vector at each point of the surface.

Definition

We can define singular points, indices of singular points the same way we defined these notions on the plane.

Note: not clear how to define index of a curve.

Vector fields on a sphere

Theorem

Suppose we are given a vector field on a sphere with finitely many singular points. Then the sum of indices of all singular points equals 2.

Corollary

Any vector field on a sphere has a singular point.

Corollary

You can't comb a hairy ball. In particular, there always will be at least one hair perpendicular to the surface.

Main theorem

Theorem (Poincare-Hopf)

If Σ is an orientable surface with no boundary, an v is a vector field on it, then sum of indices of all singular points equals to $\chi(\Sigma)$.

Sketchy proof:

We will argue by induction.

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Sketchy proof:

We will argue by induction.

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Suppose Σ has genus g. We can cut Σ as follows:

We can always make the cut along a curve with no singular points on it.

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We need to define the vector field on the two caps.

We can assume that caps are **discs** with the vector field being defined on their boundaries.

Define vector field on those discs using homothety with center at the discs' center.

Sum of the indices of singular points on the two caps is 2.

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Done.

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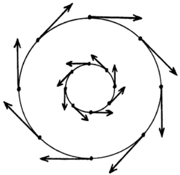
Vector field **with no singular points** *defined on the boundary of an annulus can be extended to the whole annulus* **iff** *indices of the boundary circles are equal.*



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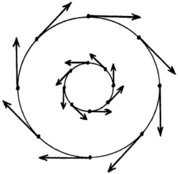

Proof:

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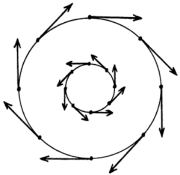

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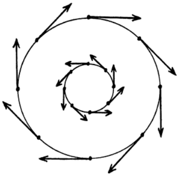

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So the indices of the circles are equal.

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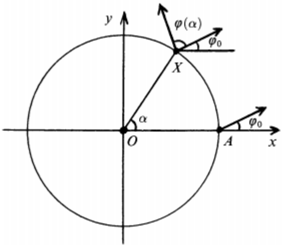
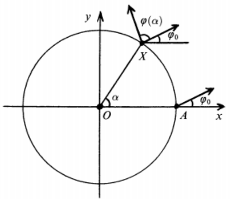
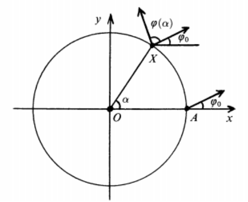
For any vector field u on a circle S^1 let's build a function ϕ as follows. Any point on S^1 is determined by its angle α. Then $\phi(\alpha)$ is the angle between the two vectors u_0 and u_α.

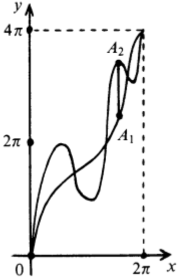




So $\phi(0) = 0$ and $\phi(2\pi) = 2k\pi$, k is the index of the circle.



So $\phi(0) = 0$ and $\phi(2\pi) = 2k\pi$, k is the index of the circle. Construct such graphs for the two circles making the boundary of the annulus. They will look something like that:



To define any (non-vanishing) vector field u of index k on a circle, it is enough so say what is $u(0)$, define a function ϕ satisfying $\phi(0) = 0$ and $\phi(2\pi) = 2k\pi$, and define a function $l(\alpha)$ which will tell the length of the vector at angle α.

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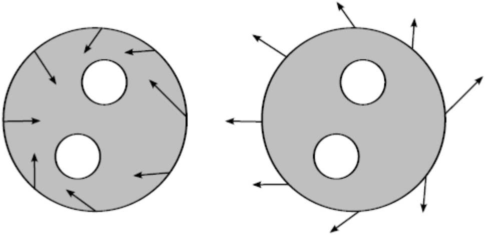
This defines the required vector field.

Alternative proof of the Poincare-Hopf theorem

Cut the surface into two pieces D_1 and D_2 as follows:



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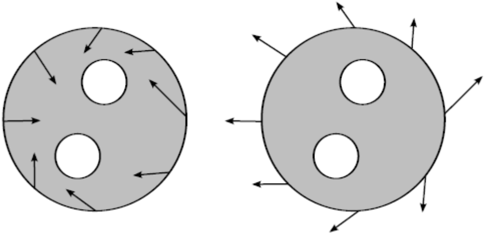

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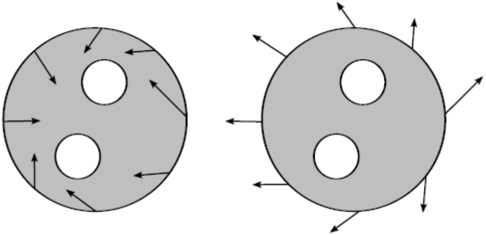

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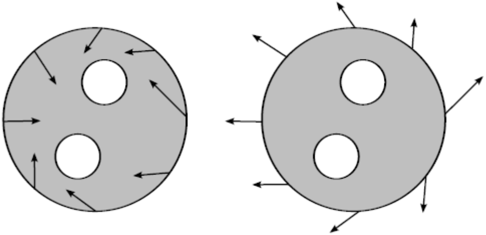

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Orient γ_0 counter-clockwise and all other γ_i's clockwise.

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$$\sum_{v_j(x)=0, x \in D_j} ind(x) = \sum_{i=0}^{g} ind_{v_j}(\gamma_i)$$

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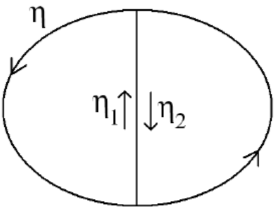
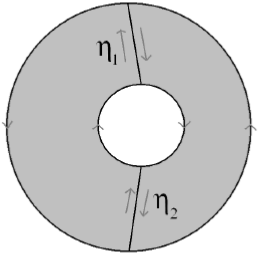
Idea: we decompose each D_j by curves, so that the regions they bound have only one singular point. Then the formula is easy.





Results of the previous slide give that

$$\sum_{i=0}^{g}(ind_{v_1}(\gamma_i) + ind_{v_2}(\gamma_i)) = \sum_{x\in\Sigma,\text{ singular}} ind(x)$$

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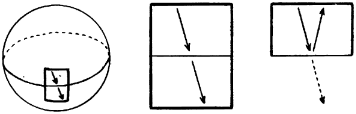
Any point on γ_0 is defined by its angle α. Vectors $v_1(\alpha)$ and $v_2(\alpha)$ are reflections of each other w.r.t. the tangent line to the circle at the point α.



(a) (b) (c)

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We are done.

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That's sad...

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